A RELATION BETWEEN SEMI-GROUPS AND SEQUENCES OF APPROXIMATION OPERATORS

BY

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1. INTRODUCTION

Let f be a twice continuously differentiable real function defined on the closed interval $[0, 1]$ of the real axis and let $B_n f$ $(n=1, 2, \ldots)$ be the n-th Bernstein polynomial associated with the function f. Then from a classical theorem of VORONOVSKAJA [5] we know that for each $x \in [0, 1]$ we have

$$\lim_{n\to\infty} n\{(B_n f)(x) - f(x)\} = \frac{x(1-x)}{2} D^2 f(x), \tag{1.1}$$

where $D^2 f$ denotes the second derivative of f.

Now we consider the right hand side as an operator. To be precise, we define the linear operator A defined on the space of all twice continuously differentiable functions f on $[0, 1]$ by

$$(Af)(x) = \frac{x(1-x)}{2} D^2 f(x). \tag{1.2}$$

Then the smallest closed extention of the linear operator A can be interpreted as an infinitesimal generator of a strongly continuous contraction semi-group of class (C_0) defined on the Banach space $C([0, 1])$. For the analytic theory of semi-groups we refer to the well-readable chapter VIII of DUNFORD and SCHWARTZ [1] or the somewhat more abstract chapter IX of YOSIDA [6]. In this paper we deal with a more general case i.e.; we investigate the possibility to associate certain infinitesimal generators of contraction semi-groups with differential expressions appearing in generalizations of the theorem of Voronovskaja. See for the main result theorem 3.

Perhaps our results may be of importance to those interested mainly in the question whenever a differential operator is the infinitesimal generator of a contraction semi-group.

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2. The theorem of Mamedov

In this section we formulate a theorem of MAMEDOV [3], which is in fact a generalization of the theorem of Voronovskaja.

However first we define.

DEFINITION 1. Let $C(\bar{I})$ be a non-empty bounded open interval of the real axis. We denote by $C(\bar{I})$ the set of all continuous real-valued functions defined on the closure $\bar{I}$ of I. By $C^k(\bar{I})$ $(k=1, 2, \ldots; \infty)$ we denote the subset of $C(\bar{I})$ of all functions which are k-times continuously differentiable on $\bar{I}$. $C_0^k(I)$ $(k=1, 2, \ldots; \infty)$ is the set of all real-valued k-times continuously differentiable functions defined on I and which have compact support in I.

Now we have the following theorem.

THEOREM 1. *Let I be a non-empty bounded open interval of the real axis and let p_k $(k=0, 1, 2)$ be the function given on $\bar{I}$ by $p_k(t)=t^k$. Moreover; let (L_n) $(n=1, 2, \ldots)$ be a sequence of positive linear operators $L_n\colon C(\bar{I}) \to C(\bar{I})$ with the property that for each $x \in \bar{I}$ we have*

i)
$$(L_n p_0)(x)=1+o\left(\frac{1}{\varphi(n)}\right)$$

$$(L_n p_1)(x)=x+\frac{\psi_1(x)}{\varphi(n)}+o\left(\frac{1}{\varphi(n)}\right) \tag{2.1}$$

$$(L_n p_2)(x)=x^2+\frac{\psi_2(x)}{\varphi(n)}+o\left(\frac{1}{\varphi(n)}\right),$$

where φ is independent of x, $\varphi(n)\neq 0$ for each n and $\lim_{n\to\infty}\varphi(n)=\infty$.
ii) *There exists an even number $r_0>2$ such that*

$$(L_n q_{r_0,x})(x)=o\left(\frac{1}{\varphi(n)}\right), \tag{2.2}$$

where $q_{r,x}(r>0, x\in\bar{I})$ is defined by $q_{r,x}(t)=(t-x)^r$.

ASSERTION: *For every function $f\in C^2(\bar{I})$ and $x\in\bar{I}$ we have*

$$(L_n f)(x)-f(x)=\frac{1}{\varphi(n)}\{\alpha(x)D^2f(x)+\beta(x)Df(x)\}+o\left(\frac{1}{\varphi(n)}\right), \tag{2.3}$$

where Df respectively D^2f are the first and second derivative of f and

$$\alpha(x)=\tfrac{1}{2}\psi_2(x)-x\psi_1(x),\ \beta(x)=\psi_1(x). \tag{2.4}$$

For a proof see theorem I of the paper of SIKKEMA [4].

It has to be noted that $\alpha(x)\geq 0$ for each $x\in\bar{I}$. This follows from the fact that

$$\alpha(x)=2\lim_{n\to\infty}\varphi(n)(L_n q_{2,x})(x), \tag{2.5}$$

which can be easily proved.

3. A relation between Mamedov's theorem and semi-groups

For the sake of completeness we shall formulate the famous theorem of Hille-Yosida (for contraction semi-groups on Banach spaces of class (C_0)), which we need here.

THEOREM 2. *Let X be a Banach space and let A be a linear operator with domain $D(A)\subset X$ and range $R(A)\subset X$. Then A is an infinitesimal generator of a strongly continuous contraction semi-group of class (C_0) if and only if $D(A)$ is dense in X and there exists a $\lambda_0>0$ such that for all $\lambda\geqq\lambda_0$ the resolvent $R(\lambda; A)=(\lambda-A)^{-1}$ is defined as an operator on X and satisfies $\|\lambda(\lambda-A)^{-1}\|\leqq 1$.*

PROOF. See e.g. YOSIDA [6], p. 248, 249 or DUNFORD and SCHWARTZ [1], p. 624–626.

Now we arrive at the main result of this paper.

THEOREM 3. *Let (L_n) $(n=1, 2, \ldots)$ be a sequence of operators, satisfying the conditions of theorem 1 and let A be the linear operator defined on $C^2(\bar{I})$, which is given by*

$$Af=\alpha D^2 f+\beta Df, \tag{3.1}$$

where α and β are defined in (2.4). Suppose in addition that

iii) a. $\alpha\in C^\infty(\bar{I})$, $\beta\in C^\infty(\bar{I})$.

b. *$\alpha(x)>0$ when $x\in I$ and $\alpha(x)=0$ when x belongs to the boundary ∂I of I (i.e. when x is an endpoint of $\bar{I}$).*

c. *α^{-1} is not integrable over neighbourhoods of endpoints of $\bar{I}$.*

d. *$\beta\cdot\alpha^{-\frac{1}{2}}$ is bounded on I.*

ASSERTIONS: *The operator A is closable in $C(\bar{I})$ provided with the topology of uniform convergence on $\bar{I}$ and the smallest closed extension $\bar{A}$ of A is an infinitesimal generator of a strongly continuous contraction semi-group $(T_t)(t\geqq 0)$ of class (C_0) on $C(\bar{I})$.*

PROOF. We prove this theorem in four steps the last one of which is established in section 6.

STEP 1. *The linear operator A is closable.* It is known that A is closable if and only if for each sequence (f_n) $(n=1, 2, \ldots)$ in $D(A)$ with $\lim_{n\to\infty} f_n=0$ and $\lim_{n\to\infty} Af_n=g\in C(\bar{I})$ it follows that $g=0$. (See e.g. YOSIDA [6], p. 77, 78). Now suppose that $f_n \in C^2(\bar{I})$, $\lim_{n\to\infty} f_n = 0$ and $\lim_{n\to\infty} Af_n = g \in C(\bar{I})$ uniformly on $\bar{I}$. Then we have for any $\varphi \in C^2(\bar{I})$ with compact support in I

$$\int_I (Af_n)\varphi dx = \int_I f_n(A^t\varphi)dx, \tag{3.2}$$

where A^t is the transposed of the differential expression A. By taking limits at both sides of equation (3.2) we get: $\int_I g\varphi dx = 0$ for each $\varphi \in C^2(\bar{I})$ with compact support in I. It follows that g is identically equal to zero on $\bar{I}$.

Step 2. *We have for each $f \in C^2(\bar{I})$, $x \in \bar{I}$ and $\lambda > 0$ the inequalities*

$$\min_{x\in\bar{I}} \{\lambda f(x) - (Af)(x)\} \leq \lambda f(x) \leq \max_{x\in\bar{I}} \{\lambda f(x) - (Af)(x)\}. \tag{3.3}$$

From the condition (iii) it follows that Af vanishes on the boundary ∂I. Hence the second part of inequality (3.3) is valid in case f attains its maximum at a point on the boundary. In case f attains its maximum at an interior point x_0 we have

$$\max_{x\in\bar{I}} \{\lambda f(x) - (Af)(x)\} \geq \lambda f(x_0) - (Af)(x_0) =$$
$$= \lambda f(x_0) - \alpha(x_0)D^2 f(x_0) - \beta(x_0)Df(x_0) \geq \lambda f(x_0)$$

since in this case at this point we have $Df(x_0) = 0$ and $D^2 f(x_0) \leq 0$.
We can prove the first part of inequality (3.3) in similar way.

Step 3. *We apply the theorem of Hille-Yosida.* Evidently, the linear manifold $C^2(\bar{I})$ is dense in $C(\bar{I})$. So the only thing we have to prove is that there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ $(\lambda - \bar{A})^{-1}$ exists with domain $C(\bar{I})$ and satisfies $\|\lambda(\lambda - \bar{A})\| \leq 1$.

From step 2 it follows that $\lambda - A$ is one-to-one on $C^2(\bar{I})$ and $\|\lambda f\| \leq \leq \|(\lambda - A)f\|$ when $f \in C^2(\bar{I})$ and $\lambda > 0$. Now if we suppose for a moment that there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the range $R(\overline{\lambda - A})$ of $\overline{\lambda - A}$ is dense in $C(\bar{I})$ it follows from the fact that $\lambda - \bar{A} = \overline{\lambda - A}$ and the continuity of the norm that the range $R(\lambda - \bar{A})$ equals $C(\bar{I})$ and $\|\lambda f\| \leq \|(\lambda - \bar{A})f\|$ when f belongs to the domain $D(\bar{A})$ of $\bar{A}$. Thus $(\lambda - \bar{A})^{-1}$ exists as a continuous linear operator of $C(\bar{I})$ into $C(\bar{I})$ such that $\|\lambda(\lambda - \bar{A})^{-1}\| \leq 1$. Then the proof would be finished.

Hence what remains to be proved of theorem 3 is the existence of a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the range $R(\overline{\lambda - A})$ is dense in $C(\bar{I})$. This we shall do in section 6.

4. Some special Hilbert spaces

In this section we deal with some special Hilbert spaces, which shall be used in the next section in the treatment of differential operators. In fact these spaces are weighted one-dimensional Sobolev spaces.

In order to make applications of spectral theory to the differential operators considered by us more simple we shall deal in this and the next section as contrasted with what precedes with complex-valued functions. Definitions concerning sets of functions are changed in an obvious way.

DEFINITION 2. $H(I, \alpha)$ is the space of complex-valued functions which are square-integrable over the interval I with respect to the measure dx/α. Here α and I are the same as in theorem 3. This space is normed by

$$\|u\|_H = \left(\int_I |u|^2 \frac{dx}{\alpha} \right)^{\frac{1}{2}} \tag{4.1}$$

REMARK: $H(I, \alpha)$ is a Hilbert space with the inner product

$$(u, v)_H = \int_I u\bar{v} \frac{dx}{\alpha} \tag{4.2}$$

and contains $C_0^\infty(I)$ as a dense subset.

DEFINITION 3. $V(I, \alpha)$ is the space of all $f \in H(I, \alpha)$ with the property that its distributional derivative Df is square-integrable over I with respect to the ordinary Lebesgue measure. This space is normed by

$$\|u\|_V = (\int_I |u|^2 \frac{dx}{\alpha} + \int_I |Du|^2 dx)^{\frac{1}{2}}. \tag{4.3}$$

REMARK: $V(I, \alpha)$ is a Hilbert space with the inner product

$$(u, v)_V = \int_I u\bar{v} \frac{dx}{\alpha} + \int_I (Du)(\overline{Dv}) dx \tag{4.4}$$

and we have:

LEMMA 1. *$C_0^\infty(I)$ is dense in $V(I, \alpha)$.*

PROOF. Let u belong to the orthogonal complement of $C_0^\infty(I)$ in $V(I, \alpha)$. This means

$$\int_I u\bar{\varphi} \frac{dx}{\alpha} + \int_I (Du)(\overline{D\varphi}) dx = 0 \tag{4.5}$$

for all $\varphi \in C_0^\infty(I)$. It follows by integration by parts that

$$\int_I u \left\{ \overline{D^2\varphi - \frac{\varphi}{\alpha}} \right\} dx = 0. \tag{4.6}$$

Thus u is a weak solution on I of the equation

$$\alpha D^2 u - u = 0. \tag{4.7}$$

Since α is a smooth function on I it follows that $u \in C^\infty(I)$ and that u is an ordinary solution of (4.7) on I (See e.g. DUNFORD and SCHWARTZ [2], p. 1291). Since α^{-1} is not integrable with respect to the Lebesgue measure over neighbourhoods of endpoints of $\bar{I}$ and members of $V(I,\alpha)$ are uniformly continuous and bounded on I it follows that u can be extended to a continuous function $\tilde{u}$ defined on $\bar{I}$, which vanishes at the boundary ∂I.

Now suppose that u is not identically equal to zero. Then there exists a point $x_0 \in I$, where u attains a maximum or a minimum not equal to zero. At this point x_0 we have $(D^2u)(x_0)\cdot u(x_0) \leq 0$, which is in contradiction with (4.7) by the positivity of α on I. This completes the proof.

Remark. We observe that $H(I,\alpha)$ and $V(I,\alpha)$ are two Hilbert spaces such that $V(I,\alpha)$ is a dense subset of $H(I,\alpha)$ with continuous injection.

5. Integration of a differential equation

Consider the differential expression

$$A_\lambda = \lambda - \alpha D^2 - \beta D \tag{5.1}$$

on I, where I, α and β are the same as in theorem 3; λ is a complex number.

Suppose that $u, g \in C_0^\infty(I)$ are related by

$$A_\lambda u = g. \tag{5.2}$$

Evidently, this relation is equivalent to: For each $v \in C_0^\infty(I)$

$$(A_\lambda u, v)_H = (g, v)_H \tag{5.3}$$

holds. By writing this out we get

$$\lambda \int_I u\bar{v}\frac{dx}{\alpha} - \int_I (D^2u)\bar{v}dx - \int_I \beta(Du)\bar{v}\frac{dx}{\alpha} = \int_I g\bar{v}\frac{dx}{\alpha} \tag{5.4}$$

and then by integration by parts of the second term we obtain

$$\lambda \int_I u\bar{v}\frac{dx}{\alpha} + \int_I (Du)(\overline{Dv})dx - \int_I \beta(Du)\bar{v}\frac{dx}{\alpha} = \int_I g\bar{v}\frac{dx}{\alpha}. \tag{5.5}$$

Let $\hat{V}(I,\alpha)$ be the linear manifold $C_0^\infty(I)$ provided with the relative topology of $V(I,\alpha)$. Then we introduce the sesqui-linear form $\hat{B}_\lambda$ on $\hat{V}(I,\alpha)\times\hat{V}(I,\alpha)$ by

$$\hat{B}_\lambda(u, v) = \lambda \int_I u\bar{v}\frac{dx}{\alpha} + \int_I (Du)(\overline{Dv})dx - \int_I \beta(Du)\bar{v}\frac{dx}{\alpha}. \tag{5.6}$$

For (5.4) we may then write

$$\hat{B}_\lambda(u, v) = (g, v)_H. \tag{5.7}$$

Now we have the following estimates:

$$\left|\lambda \int_I u\bar{v}\frac{dx}{\alpha}\right| \leq |\lambda|\ \|u\|_V\|v\|_V: \tag{5.8}$$

$$\left|\int_I (Du)(\overline{Dv})dx\right| \leq \|u\|_V\|v\|_V: \tag{5.9}$$

By the condition (iii-d) of theorem 3 we see that there exists a constant $c_1 > 0$ such that

$$(5.10)\quad \begin{cases} \left|\int_I \beta(Du)\bar{v}\frac{dx}{\alpha}\right| \leqq c_1 \int_I |Du| \left|\frac{v}{\alpha^{\frac{1}{2}}}\right| dx \leqq \\ \qquad \leqq c_1\{\int_I |Du|^2 dx\}^{\frac{1}{2}} \cdot \left\{\int_I |v|^2 \frac{dx}{\alpha}\right\}^{\frac{1}{2}} \leqq c_1 \|u\|_V \|v\|_V. \end{cases}$$

Hence $\hat{B}_\lambda$ is a bounded sesqui-linear form on $\hat{V}(I,\alpha) \times \hat{V}(I,\alpha)$ and therefore, remembering that $C_0^\infty(I)$ is dense in $V(I,\alpha)$, we see that $\hat{B}_\lambda$ admits a unique bounded extension B_λ to $V(I,\alpha) \times V(I,\alpha)$.

Next we prove that B_λ is coercive for $\operatorname{Re}\lambda$ sufficiently large, i.e.; there exist numbers $\lambda_0 > 0$ and $c_2 > 0$ such that

$$(5.11)\qquad \operatorname{Re} B_\lambda(u,u) \geqq c_2 \|u\|_V^2$$

for all λ with $\operatorname{Re}\lambda \geqq \lambda_0$ and $u \in V(I,\alpha)$. From (5.10) in combination with μ the inequality

$$(5.12)\qquad |\alpha|\ |\beta| \leqq \varepsilon|\alpha|^2 + \frac{1}{\varepsilon}|\beta|^2,$$

which is valid for every $\varepsilon > 0$, we get for each $u \in C_0^\infty(I)$

$$(5.13)\qquad \left|\int_I \beta(Du)\bar{u}\frac{dx}{\alpha}\right| \leqq c_1\varepsilon \int_I |Du|^2 dx + \frac{c_1}{\varepsilon}\int_I |u|^2 \frac{dx}{\alpha}.$$

From the definition (5.6) and the inequality (5.13) we get

$$(5.14)\quad \begin{cases} \operatorname{Re}\hat{B}_\lambda(u,u) \geqq \operatorname{Re}\lambda \int_I |u|^2\frac{dx}{\alpha} + \int_I |Du|^2 dx - c_1\varepsilon\int_I |Du|^2 dx \\ \qquad - \frac{c_1}{\varepsilon}\int_I |u|^2 \frac{dx}{\alpha} = \left(\operatorname{Re}\lambda - \frac{c_1}{\varepsilon}\right)\int_I |u|^2\frac{dx}{\alpha} + (1 - c_1\varepsilon)\int_I |Du|^2 dx. \end{cases}$$

So by taking ε small enough we have proved (5.11) in case $u \in C_0^\infty(I)$. By the continuity of B_λ and the $V(I,\alpha)$-norm we have finished our proof.

Summarizing we have

THEOREM 4. *$V(I,\alpha)$ is the space of definition 3. $\hat{V}(I,\alpha)$ is the linear manifold $C_0^\infty(I)$ provided with the relative topology of $V(I,\alpha)$. $\hat{B}_\lambda$ is the sesqui-linear form on $\hat{V}(I,\alpha) \times \hat{V}(I,\alpha)$ defined in* (5.6).

Then $\hat{B}_\lambda$ is bounded and the unique extension B_λ of $\hat{B}_\lambda$ to $V(I,\alpha) \times V(I,\alpha)$ is for $\operatorname{Re}\lambda$ *sufficiently large, let say for* $\operatorname{Re}\lambda \geqq \lambda_0$, *coercive (for the definition see around* (5.11)).

ADDITIONAL STATEMENT: *There exist a family of bounded linear maps $(\mathscr{L}_\lambda)(\lambda \in C)$, $\mathscr{L}_\lambda \colon V(I,\alpha) \to V(I,\alpha)$, such that*

$$(5.15)\qquad B_\lambda(u,v) = (\mathscr{L}_\lambda u, v)_V$$

if $u, v \in V(I,\alpha)$ and $\mathscr{L}_\lambda$ is a linear isomorphism for $\operatorname{Re}\lambda \geqq \lambda_0$.

PROOF. The last assertion is a consequence of the theorem of Lax-Milgram and the fact that B_λ is coercive for $\operatorname{Re}\lambda \geqq \lambda_0$.

Let J be the canonical injection of $V(I, \alpha)$ into $H(I, \alpha)$ and J^* its adjoint defined by:

$$(Ju, v)_H = (u, J^*v)_V \tag{5.16}$$

for all $u \in V(I, \alpha)$ and $v \in H(I, \alpha)$. We know that J has dense range; hence J^* is injective. In addition it can easily be seen that the range $R(J^*)$ of J^* consists of all $w \in V(I, \alpha)$ such that the map $v \mapsto (v, w)_V$ is continuous with respect to the topology induced by the $H(I, \alpha)$-norm on $V(I, \alpha)$.

Now we obtain from (5.15) and (5.16)

$$B_\lambda(u, v) = ((J^*)^{-1} \circ \mathscr{L}_\lambda u, v)_H \tag{5.17}$$

if $v \in V(I, \alpha)$ and if u is such that $\mathscr{L}_\lambda u \in R(J^*)$.

In other words: (5.17) holds for all $v \in V(I, \alpha)$ and the set D_λ of all $u \in V(I, \alpha)$ such that the map $v \mapsto B_\lambda(u, v)$ is continuous with respect to the topology induced by the $H(I, \alpha)$-norm on $V(I, \alpha)$. Consequently, by the definition of B_λ we see that D_λ is independent of λ; we shall denote it by D. Define the linear map $L_\lambda = (J^*)^{-1} \circ \mathscr{L}_\lambda$ of D into $H(I, \alpha)$. We know that $\mathscr{L}_\lambda$ is a linear isomorphism for $\operatorname{Re}\lambda \geqq \lambda_0$. Hence L_λ has for $\operatorname{Re}\lambda \geqq \lambda_0$ as inverse the bounded linear operator $G_\lambda = \mathscr{L}_\lambda^{-1} \circ J^*$ with domain $H(I, \alpha)$ and range D.

It follows that for $\operatorname{Re}\lambda \geqq \lambda_0$ the system,

$$B_\lambda(u, v) = (g, v)_H \tag{5.18}$$

for all $v \in V(I, \alpha)$, has for each $g \in H(I, \alpha)$ a unique solution $G_\lambda g$ in $V(I, \alpha)$ which belongs to D.

In the following theorem we characterize the operator L_λ and the set D.

THEOREM 5. i) *$L_\lambda u = A_\lambda u$ for all $u \in D$ and $\lambda \in C$, where*

$$D = \{u \in V(I, \alpha) | A_0 u \in H(I, \alpha)\}.$$

Here A_λ is the operator defined in (5.1), *to be applied in distributional sense.*

ii) *D contains the set $C_0^\infty(I)$.*

REMARK. The remark succeeding definition 2, lemma 1 (both in section 4) and ii) imply that D is dense in $H(I, \alpha)$ and in $V(I, \alpha)$.

PROOF i). From (5.6) it follows that for all $u, \varphi \in C_0^\infty(I)$ we have

$$\hat{B}_\lambda(u, \varphi) = (A_\lambda u, \varphi)_H = \int_I (A_\lambda u)\bar{\varphi}\frac{dx}{\alpha} = \int_I u A_\lambda^t\left(\frac{\bar{\varphi}}{\alpha}\right) dx, \tag{5.19}$$

when A_λ^t is the transposed of A_λ.

Since $\hat{B}_\lambda$ admits a unique bounded extension B_λ we see that

$$B_\lambda(u, \varphi) = \int_I u A_\lambda^\dagger\left(\frac{\bar{\varphi}}{\alpha}\right) dx = \int_I (A_\lambda u)\bar{\varphi}\,\frac{dx}{\alpha} \tag{5.20}$$

for all $u \in V(I, \alpha)$ and all $\varphi \in C_0^\infty(I)$, where now A_λ has to be applied in distributional sense.

In case $u \in D$ and $\varphi \in C_0^\infty(I)$ we can write also

$$B_\lambda(u, \varphi) = (L_\lambda u, \varphi) = \int_I (L_\lambda u)\bar{\varphi}\,\frac{dx}{\alpha}. \tag{5.21}$$

Comparing (5.20) with (5.21) we see that if $u \in D$ it follows that $L_\lambda u = A_\lambda u$ and $L_\lambda u \in H(I, \alpha)$.

On the other hand if $A_0 u \in H(I, \alpha)$ we have

$$B_0(u, \varphi) = \int_I (A_0 u)\bar{\varphi}\,\frac{dx}{\alpha} = (A_0 u, \varphi)_H \tag{5.22}$$

for all $\varphi \in C_0^\infty(I)$. Now B_0 is continuous on $V(I, \alpha) \times V(I, \alpha)$ and $C_0^\infty(I)$ is dense in $V(I, \alpha)$. Thus

$$B_0(u, v) = (A_0 u, v)_H \tag{5.23}$$

for all $v \in V(I, \alpha)$. Hence we conclude that the map $v \mapsto B_0(u, v)$ is continuous on $V(I, \alpha)$. Thus $u \in D_0 = D$.

Proof ii). This follows immediately from the definition of A_λ in (5.1).

6. End of the proof of theorem 3

By the results of section 5 we now have

Corollary (*step* 4). *Let A be the operator defined in* (3.1). *Then there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the range $R(\overline{\lambda - A})$ of the operator $\overline{\lambda - A}$ is dense in $C(\bar{I})$.*

Proof. Let λ_0 be the λ_0 of section 5 (around (5.11)). Let $\lambda \geq \lambda_0$ and λ be fixed. Let M be the linear manifold spanned by the two elements f_1, f_2 given by

$$f_1(t) = \lambda t - \beta(t),\ f_2(t) = 1. \tag{6.1}$$

Then we can split $C(\bar{I})$ up into the direct sum

$$C(\bar{I}) = M \oplus C_v(\bar{I}), \tag{6.2}$$

where $C_v(\bar{I})$ is the space of all $f \in C(\bar{I})$ which are zero at the boundary of $\bar{I}$. We remark that $C_0^\infty(I)$ is a dense subset of $C_v(\bar{I})$.

Since M is the image under $\lambda - A$ of the polynomials of degree at most one and $C_0^\infty(I)$ is contained in the range of $\overline{\lambda - A}$ it follows by the linearity of the operator $\overline{\lambda - A}$ that the range of $\overline{\lambda - A}$ is dense in $C(\bar{I})$. Thus the proof of the corollary and with it that of theorem 3 is finished.

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